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# A family of second Lie algebra structures for symmetries of a dispersionless Boussinesq system

**Arthemy V Kiselev and Johan W van de Leur**

Mathematical Institute, University of Utrecht, PO Box 80.010, 3508 TA Utrecht, The Netherlands

E-mail: A.V.Kiselev@uu.nl and J.W.vandeLeur@uu.nl

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**Abstract**
For the 3-component dispersionless Boussinesq-type system, we construct two compatible nontrivial finite deformations for the Lie algebra structure in the symmetry algebra.

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In this paper, we construct a two-parametric family of nontrivial finite deformations for the Lie bracket in the algebra of symmetries for the 3-component dispersionless Boussinesq system of hydrodynamic type [1–3]

$$\mathcal{E} = \{u_t = ww_x + v_x,\ v_t = -uw_x - 3u_x w,\ w_t = u_x\}. \qquad (1)$$

This system is obtained by reduction from the dispersionless Kadomtsev–Petviashvili (dKP) equation; its integrability by the generalized hodograph transformation is discussed in [1].

First, we note that the image of a previously known [3] self-adjoint Noether operator $A_0\colon \operatorname{cosym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ for (1) is closed w.r.t. the commutation. Hence, this operator and the bi-Hamiltonian pair $(\hat{A}_1, \hat{A}_2)$ for (1), see [2], transfer the standard bracket $[\,,\,]$ in $\operatorname{sym}\mathcal{E}$ to the Lie algebra structures on their domain. We prove that the three new brackets are compatible.

The Noether operator $A_0$ is invertible on an open dense subset of $\mathcal{E}$. This yields two recursion operators $R_i = \hat{A}_i \circ A_0^{-1}\colon \operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$. The images of $R_i$ are again closed w.r.t. the commutation, and this property is retained by their arbitrary linear combinations. Using the 'chain rule' formula (8) for the bi-differential brackets on domains of the operators $\hat{A}_i$ and $R_i$, we calculate the second Lie algebra structures $[\,,\,]_{R_i}$ on $\operatorname{sym}\mathcal{E}$.

All notions and constructions are standard [4, 5]. We follow the notation of [3, 5]. We stress that the concept of linear compatible differential operators with involutive images, which we present here, can be applied to the study of other integrable systems with or without dispersion (e.g., see [6]).

**Remark 1.** Let $M$ be a smooth finite-dimensional orientable real manifold. The construction of *trivial infinitesimal* deformations

$$[x, y]_{\mathrm{N}} := \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{e}^{-\lambda \mathrm{N}}[\mathrm{e}^{\lambda \mathrm{N}}(x), \mathrm{e}^{\lambda \mathrm{N}}(y)] = [\mathrm{N}x, y] + [x, \mathrm{N}y] - \mathrm{N}([x, y])$$

of the standard Lie algebra structure $[\,,]$ on the tangent bundle, $x, y \in \Gamma(TM)$, is well developed in the literature [7]: if the Nijenhuis torsion $[\mathrm{N}x, \mathrm{N}y] - \mathrm{N}([x, y]_{\mathrm{N}})$ for an endomorphism $\mathrm{N}\colon \Gamma(TM) \to \Gamma(TM)$ vanishes, then the Lie brackets $[\,,]_{\mathrm{N}^k}$ obtained by iterations $\mathrm{N}^k$ of the Nijenhuis recursion N are compatible (their linear combinations are Lie algebra structures as well). Suppose further that $M$ is equipped with a Poisson bi-vector $\mathcal{P} \in \Gamma(\bigwedge^2(TM))$. If the Nijenhuis and Poisson structures $(\mathrm{N}, \mathcal{P})$ satisfy two compatibility conditions [7], then they generate an infinite hierarchy of compatible Poisson structures $\mathrm{N}^k \circ \mathcal{P}, k \geqslant 0$. The concept of Poisson–Nijenhuis structures admits a straightforward generalization [8] for the infinite jet bundles over smooth manifolds and for infinite-dimensional integrable systems of PDE; see [4, 9, 10] for details.

In contrast, in this paper we construct two *nontrivial finite* deformations $[\,,]_{R_i}$ of the standard Lie bracket $[\,,]$ on the symmetry algebra $\operatorname{sym}\mathcal{E}$ for (1). We shall use two local recursion operators $R_i, i = 1, 2$, whose images are closed w.r.t. the commutation. Thence, we obtain the new bracket $[\,,]_{R_i}$ through

$$[R_i\varphi_1, R_i\varphi_2] = R_i\big([\varphi_1, \varphi_2]_{R_i}\big) \qquad \text{for any } \varphi_1, \varphi_2 \in \operatorname{sym}\mathcal{E}. \tag{2}$$

The construction consists of two steps.

## 1. Involutive distributions of operator-valued fields and the linear compatibility of operators

First, let a linear operator $\square$ in total derivatives be either a recursion $\operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ for an evolutionary system $\mathcal{E}$ or a Noether operator $\operatorname{cosym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ whose arguments, the cosymmetries[1] $\psi$, belong to the kernel of the adjoint linearization $\ell^*_{\mathcal{E}}(\psi) = 0$ for $\mathcal{E} = \{u_t = \boldsymbol{f}\}$; see [4, 5]. For example, all Hamiltonian operators for $\mathcal{E}$ are Noether.

Suppose further that the image of $\square$ is closed w.r.t. the commutation in $\operatorname{sym}\mathcal{E}$: $[\operatorname{im}\square, \operatorname{im}\square] \subseteq \operatorname{im}\square$. The Lie algebra structure $[\,,]|_{\operatorname{im} A}$ is transferred by $\square$ onto the quotient $\Omega = \operatorname{dom}\square/\ker\square$:

$$[\square(\phi'), \square(\phi'')] = \square([\phi', \phi'']_{\square}), \qquad \phi', \phi'' \in \Omega.$$

By the Leibniz rule, two pairs of summands appear in the bracket of the evolutionary vector fields $\partial_{\square(\phi')}$ and $\partial_{\square(\phi'')}$, which are of form $\partial_{\varphi} = \sum_{|\sigma|\geqslant 0} D_\sigma(\varphi)\cdot \partial/\partial u_\sigma$,

$$[\square(\phi'), \square(\phi'')] = \square(\partial_{\square(\phi')}(\phi'') - \partial_{\square(\phi'')}(\phi')) + (\partial_{\square(\phi')}(\square)(\phi'') - \partial_{\square(\phi'')}(\square)(\phi')).$$

In the first summand, we have used the permutability of evolutionary derivations and total derivatives. The second pair of summands must hit the image of $\square$ by the assumption of commutation closure. Therefore, the bracket $[\phi', \phi'']_{\square}$ equals

$$[\phi', \phi'']_{\square} = \partial_{\square(\phi')}(\phi'') - \partial_{\square(\phi'')}(\phi') + \{\!\{\phi', \phi''\}\!\}_{\square}. \tag{3}$$

[1] We label the evolution equations upon $u^1, \ldots, u^m$ in the system $\mathcal{E}$ with the same variables $u^i$ that occur in the left-hand sides. With such convention, the cosymmetries $\psi$ that originate from the currents $\eta = \rho\,\mathrm{d}x + \cdots$ are equal to the variational derivatives $\delta\rho/\delta u$ of the conserved densities $\rho$. By extending the transformation rule for $\delta\rho/\delta u$ under reparametrizations of $u^i$ that preserve the evolutionary form of $\mathcal{E}$ onto the entire domain of $\square$, we obtain the transformation law for Noether operators.

It contains the two standard summands and the skew-symmetric bilinear bracket $\{\!\{,\}\!\}_\Box$. Unlike $[,]_\Box$, its component $\{\!\{,\}\!\}_\Box$ does not generally satisfy the Jacobi identity and it may not be a cocycle for the Lie algebra $(\Omega, [,]_\Box)$. For Hamiltonian operators $\hat{A}$, the bracket $\{\!\{,\}\!\}_{\hat{A}}$ is given by (5); see lemma 1. The decomposition (3) of the bracket $[,]_\Box$ that calculates the commutation relations for higher symmetries $\varphi = \Box(\phi)$ of the scalar Liouville equation was obtained in [11], cf [6].

**Example 1.** A self-adjoint (hence non-Hamiltonian[2]) zero-order Noether operator $A_0$ for (1) was found in [3]:

$$A_0 = \begin{pmatrix} ww_x + v_x & -3u_x w - uw_x & u_x \\ -3u_x w - uw_x & -3w^2 w_x - 4v_x w - uu_x & v_x \\ u_x & v_x & w_x \end{pmatrix}. \tag{4}$$

We see that its image in $\operatorname{sym}\mathcal{E}$ is involutive; the components of the arising bracket $\{\!\{\boldsymbol{p}, \boldsymbol{q}\}\!\}_{A_0}$ with $\boldsymbol{p}, \boldsymbol{q} \in \operatorname{cosym}\mathcal{E}$ are

$$\begin{aligned}
\{\!\{\boldsymbol{p}, \boldsymbol{q}\}\!\}^u_{A_0} &= p^w_x q^u - p^u q^w_x + 3w\big(p^u q^v_x - p^v_x q^u\big) + 3w\big(p^v q^u_x - p^u_x q^v\big) \\
&\quad + p^u_x q^w - p^w q^u_x + 2w_x\big(p^u q^v - p^v q^u\big) + u\big(p^v q^v_x - p^v_x q^v\big), \\
\{\!\{\boldsymbol{p}, \boldsymbol{q}\}\!\}^v_{A_0} &= p^u_x q^u - p^u q^u_x + 4w\big(p^v q^v_x - p^v_x q^v\big) + p^v_x q^w - p^w q^v_x + p^w_x q^v - p^v q^w_x, \\
\{\!\{\boldsymbol{p}, \boldsymbol{q}\}\!\}^w_{A_0} &= u\big(p^u q^v_x - p^v_x q^u\big) + 3w^2\big(p^v q^v_x - p^v_x q^v\big) + 2u_x\big(p^v q^u - p^u q^v\big) \\
&\quad + w\big(p^u_x q^u - p^u q^u_x\big) + u\big(p^v q^u_x - p^u_x q^v\big) + p^w_x q^w - p^w q^w_x.
\end{aligned}$$

**Lemma 1** ([5, p 130])**.** *The image of any Hamiltonian operator* $\hat{A} = \big\|\sum_{|\tau|\geqslant 0} A^{ij}_\tau \cdot D_\tau\big\|$ *is closed w.r.t. the commutation. The kth component* ($1 \leqslant k \leqslant m$) *of the bracket* $\{\!\{,\}\!\}_{\hat{A}}$ *on its domain equals*

$$\{\!\{\phi', \phi''\}\!\}^k_{\hat{A}} = \sum_{|\sigma|\geqslant 0} \sum_{i=1}^{m} (-1)^{|\sigma|} \left( D_\sigma \circ \left[ \sum_{|\tau|\geqslant 0} \sum_{j=1}^{m} D_\tau(\phi'_j) \cdot \frac{\partial A^{ij}_\tau}{\partial u^k_\sigma} \right] \right) (\phi''_i). \tag{5}$$

The proof of lemma 1 essentially amounts to an evaluation of the preimage of the expression $\partial_{A(\phi')}(A)(\phi'') - \partial_{A(\phi'')}(A)(\phi')$ under the mapping $A$ (cf (3) with $\Box = A$). This is achieved via formula (A.1) from theorem 6, which exprimes the fact that a linear skew-adjoint total differential operator $A$ is Hamiltonian if and only if the Poisson bracket $\{,\}_A$ satisfies the Jacobi identity or, equivalently, the Schouten bracket $[\![A, A]\!]$ vanishes; see appendix A for details.

**Example 2.** The pair $(\hat{A}_1, \hat{A}_2)$ of compatible Hamiltonian operators for (1) was obtained in [2]:

$$\hat{A}_1 = \begin{pmatrix} D_x & 0 & 0 \\ 0 & -4wD_x - 2w_x & D_x \\ 0 & D_x & 0 \end{pmatrix}, \tag{6}$$

$$\hat{A}_2 = \begin{pmatrix} (2w^2 + 4v)D_x + 2(ww_x + v_x) & -11uwD_x - (5u_x w + 9uw_x) & 3uD_x + u_x \\ -11uwD_x - 6u_x w - 2uw_x & 2hD_x + h_x & 4vD_x + v_x \\ 3uD_x + 2u_x & 4vD_x + 3v_x & 2wD_x + w_x \end{pmatrix}, \tag{7}$$

[2] In this paper, Hamiltonian operators are indicated with the 'hat' sign: $\hat{A}$, $\hat{A}_1$ and $\hat{A}_2$.

where we put $h = -\big(\frac{3}{2}u^2 + 8vw + 3w^3\big)$. The components of the brackets $\{\{,\}\}_{\hat{A}_i}$ are given by (5): for any $\boldsymbol{p}, \boldsymbol{q} \in \operatorname{cosym}\mathcal{E}$ they equal, respectively,

$$\{\{,\}\}^u_{\hat{A}_1} = \{\{,\}\}^v_{\hat{A}_1} = 0, \qquad \{\{\boldsymbol{p},\boldsymbol{q}\}\}^w_{\hat{A}_1} = 2\big(p^v q^v_x - p^v_x q^v\big)$$

and

$$\begin{aligned}
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^u_{\hat{A}_2} &= 2\big(p^w_x q^u - p^u q^w_x\big) + 6w\big(p^u q^v_x - p^v_x q^u\big) + 5w\big(p^v q^u_x - p^u_x q^v\big)\\
&\quad + p^u_x q^w - p^w q^u_x + 4w_x(p^u q^v - p^v q^u) + 3u\big(p^v q^v_x - p^v_x q^v\big),\\
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^v_{\hat{A}_2} &= 2\big(p^u_x q^u - p^u q^u_x\big) + 8w\big(p^v q^v_x - p^v_x q^v\big) + p^v_x q^w - p^w q^v_x + 3\big(p^w_x q^v - p^v q^w_x\big),\\
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^w_{\hat{A}_2} &= 2u\big(p^u q^v_x - p^v_x q^u\big) + \big(8v + 9w^2\big)\cdot\big(p^v q^v_x - p^v_x q^v\big) + 4u_x(p^v q^u - p^u q^v)\\
&\quad + 2w\big(p^u_x q^u - p^u q^u_x\big) + 9u\big(p^v q^u_x - p^u_x q^v\big) + p^w_x q^w - p^w q^w_x.
\end{aligned}$$

We claim that the Noether operator $A_0$, whose image in $\operatorname{sym}\mathcal{E}$ is closed w.r.t. the commutation, is compatible with $\hat{A}_1$ and $\hat{A}_2$ in the following sense.

**Definition.** *We say that $N \geqslant 2$ operators $A_i\colon (\mathrm{co})\mathrm{sym}\,\mathcal{E} \to \mathrm{sym}\,\mathcal{E}$ on $\mathcal{E}$ with a common domain and involutive images, $[\operatorname{im} A_i, \operatorname{im} A_i] \subseteq \operatorname{im} A_i$ for $1 \leqslant i \leqslant N$, are linear compatible if their linear combinations $A_{\boldsymbol{\lambda}} = \sum_{i=1}^N \lambda_i A_i$ retain the same property of involutivity for any $\boldsymbol{\lambda}$.*

**Example 3.** It can easily be checked that three Noether operators (4), (6) and (7) for system (1) are linear compatible.

We note that linear compatible Hamiltonian operators are Poisson compatible and *vice versa*, because formula (5) is linear in coefficients of $\hat{A}$.

**Theorem 2.** *The bracket $\{\{,\}\}_{A_{\boldsymbol{\lambda}}}$ on the domain of the combination $A_{\boldsymbol{\lambda}}$ of linear compatible operators $A_i$ is*

$$\{\{,\}\}_{\sum_{i=1}^N \lambda_i A_i} = \sum_{i=1}^N \lambda_i \cdot \{\{,\}\}_{A_i}.$$

*The pairwise linear compatibility implies the collective linear compatibility of $A_1, \ldots, A_N$.*

**Proof.** This is readily seen by inspecting the coefficients at $\lambda_i^2$ in the quadratic polynomials in $\lambda_i$ that appear in both sides of the equality $[A_{\boldsymbol{\lambda}}(\boldsymbol{p}), A_{\boldsymbol{\lambda}}(\boldsymbol{q})] = A_{\boldsymbol{\lambda}}([\boldsymbol{p},\boldsymbol{q}]_{A_{\boldsymbol{\lambda}}})$, here $\boldsymbol{p}, \boldsymbol{q} \in \Omega = \operatorname{dom} A_i \big/ \bigcap_{j=1}^N \ker A_j$ for any $i$. □

**Corollary 3.** *Two such operators $A, B\colon (\mathrm{co})\mathrm{sym}\,\mathcal{E} \to \mathrm{sym}\,\mathcal{E}$ on $\mathcal{E}$ with a common domain and involutive images are linear compatible iff for any $\boldsymbol{p}, \boldsymbol{q} \in \Omega$ one has*

$$[B(\boldsymbol{p}), A(\boldsymbol{q})] + [A(\boldsymbol{q}), B(\boldsymbol{p})] = A([\boldsymbol{p},\boldsymbol{q}]_B) + B([\boldsymbol{p},\boldsymbol{q}]_A),$$

*which is equivalent to the relation*

$$\partial_{A(\boldsymbol{p})}(B)(\boldsymbol{q}) + \partial_{B(\boldsymbol{p})}(A)(\boldsymbol{q}) - \partial_{A(\boldsymbol{q})}(B)(\boldsymbol{p}) - \partial_{B(\boldsymbol{q})}(A)(\boldsymbol{p}) = A(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_B) + B(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A).$$

## 2. Recursion operators and new Lie brackets

The second step in the construction of the new Lie brackets on $\operatorname{sym}\mathcal{E}$ is as follows. We note that the zero-order operator (4) has the inverse $\omega = A_0^{-1}$ on an open dense subset of $\mathcal{E}$. This yields the local recursion operators $R_i := \hat{A}_i \circ A_0 \colon \operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$. By construction, their images are closed w.r.t. the commutation $[\,,\,]$. (We note also that the images of the Hamiltonian operators $\hat{A}_i$ for $\mathcal{E}$ are generally not closed w.r.t. $[\,,\,]_{R_j}$, here $1 \leqslant i, j \leqslant 2$.) Therefore, we introduce the new Lie algebra structures $[\,,\,]_{R_i}$ on $\operatorname{sym}\mathcal{E}$ by (2). Let us summarize the result.

**Proposition 4.** *The recursion operators $R_i = \hat{A}_i \circ A_0^{-1}$, $i = 1, 2$, for system (1) are linear compatible. The new Lie algebra structures $[\,,\,]_{R_i}$ on $\operatorname{sym}\mathcal{E}$ span the two-dimensional space of compatible nontrivial finite deformations of the standard bracket $[\,,\,]$.*

The transformation rules for $R_i$ and hence for $[\,,\,]_{R_i}$ under any reparametrizations of the variables $u, v, w$, are obvious. The arising bi-differential brackets $\{\![\,,\,]\!\}_{R_i}$, see (3), are obtained from $\{\![\,,\,]\!\}_{\hat{A}_i}$ using the following 'chain rule' (cf [12]).

**Theorem 5.** *Consider linear total differential operators $A\colon (\mathrm{co})\operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ and $\omega\colon (\mathrm{co})\operatorname{sym}\mathcal{E} \to (\mathrm{co})\operatorname{sym}\mathcal{E}$ such that $\operatorname{im}\omega \subseteq \operatorname{dom}A$. If the images of the operators $A$ and $R = A \circ \omega$ are closed w.r.t. the commutation of evolutionary vector fields, then the brackets $\{\![\,,\,]\!\}_A$ and $\{\![\,,\,]\!\}_R$ are related by the formula*

$$\omega(\{\![\xi_1, \xi_2]\!\}_R) = \partial_{R(\xi_1)}(\omega)(\xi_2) - \partial_{R(\xi_2)}(\omega)(\xi_1) + \{\![\omega(\xi_1), \omega(\xi_2)]\!\}_A \tag{8}$$

*for any sections $\xi_1, \xi_2$ that belong to the domain of $\omega$.*

**Proof.** Denote $\psi_i = \omega(\xi_i)$ and $\varphi_i = A(\psi_i)$ for $i = 1, 2$. We have

$$[\varphi_1, \varphi_2] = (A \circ \omega)\big(\partial_{\varphi_1}(\xi_2) - \partial_{\varphi_2}(\xi_1) + \{\![\xi_1, \xi_2]\!\}_{A\circ\omega}\big). \tag{9}$$

On the other hand, we recall that $\psi_i = \omega(\xi_i)$ and deduce

$$\begin{aligned}[\varphi_1, \varphi_2] &= A\big(\partial_{\varphi_1}(\psi_2) - \partial_{\varphi_2}(\psi_1) + \{\![\psi_1, \psi_2]\!\}_A\big)\\ &= (A \circ \omega)\big(\partial_{\varphi_1}(\xi_2) - \partial_{\varphi_2}(\xi_1)\big) + A\big(\partial_{\varphi_1}(\omega)(\xi_2) - \partial_{\varphi_2}(\omega)(\xi_1) + \{\![\psi_1, \psi_2]\!\}_A\big).\end{aligned} \tag{10}$$

Now subtract (9) from (10). Comparing (8) and (10) yields the assertion[3]. □

The bi-differential brackets $\{\![\,,\,]\!\}_{R_i}$ for the recursions $R_i$ constructed above are completely determined by the chain rule (8) with $\omega = A_0^{-1}$:

$$\{\![\varphi_1, \varphi_2]\!\}_{R_i} = A_0\big(\partial_{R_i(\varphi_1)}(\omega)(\varphi_2) - \partial_{R_i(\varphi_2)}(\omega)(\varphi_1) + \{\![\omega(\varphi_1), \omega(\varphi_2)]\!\}_{\hat{A}_i}\big),$$

here $\varphi_1, \varphi_2 \in \operatorname{sym}\mathcal{E}$ are any symmetries of (1). The three components of each bracket $\{\![\,,\,]\!\}_{R_i}$ can be calculated explicitly. The coefficients of the skew-symmetric couplings $D_x^\alpha(\varphi_1^a) \cdot D_x^\beta(\varphi_2^b) - D_x^\beta(\varphi_1^b) \cdot D_x^\alpha(\varphi_2^a)$, $0 \leqslant \alpha + \beta \leqslant 1$, $a, b \in \{u, v, w\}$, are relatively large due to the presence of the powers $(\det A_0)^\alpha$, $1 \leqslant \alpha \leqslant 3$ in the denominators. Be that as it may, the two local recursion operators $R_i$ generate the first known examples of compatible well-defined new Lie brackets on the symmetry algebra of system (1) via the Yang–Baxter equation (2)[4].

**Remark 2.** The application of the classical $r$-matrix formalism [13] for a given Lie algebra $\mathfrak{g}$ generates Liouville integrable systems using the second Lie algebra structure $[\,,\,]_r$, where $r$

[3] The bracket $\{\![\,,\,]\!\}_A$ specifies the equivalence classes modulo $\ker A$ in $\operatorname{dom}A$ and therefore an assumption that the operator $A$ be invertible is not required for (8).

[4] The assumption (2) of the commutation closure corresponds to the degenerate case $\alpha = 0$ in $\mathrm{YB}(\alpha)$.

solves the Yang–Baxter equation YB($\alpha$). Since we aim at producing new integrable systems, it seems worthy to perform the second iteration of this generator by taking $\mathfrak{g} = \operatorname{sym}\mathcal{E}$ as the initial algebra and by finding recursion operators $R$ that specify the new brackets $[,]_R$ on $\mathfrak{g}$ and the new Poisson structures on the hierarchy of Hamiltonians for $\mathcal{E}$.

**Remark 3.** Examples of non-Hamiltonian linear operators with involutive images, and Lie brackets on their domains, are scattered in the literature (e.g., see [14] for a dispersionless set-up and a bracket of 1-forms). In particular, the continuous contractions (see [15] and references therein) by $[x, y]_\epsilon = R^{-1}(\epsilon)[R(\epsilon)x, R(\epsilon)y]$, $R: \epsilon \in (0, 1] \to GL(m)$, of the brackets $[,]$ in $m$-dimensional Lie algebras $(\Bbbk^m, [,]) \ni x, y$ are the finite-dimensional analogs, in the sense of remark 1, of the recursion differential operators with involutive images and the induced brackets (3). However, let us recall that, first, the differential order of such operators can be sufficiently high for systems with dispersion. Second, the domains and images of such operators can be formed by (co)symmetries of two *different* equations. For example, a class of higher order operators with involutive images is known for the open 2D Toda chains and the related KdV-type systems; see [6]. Involutive distributions of operator-valued evolutionary vector fields will be the subject of a subsequent publication; see [16].

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### Appendix A. The bracket $\{\![,]\!\}_{\hat{A}}$ for Hamiltonian operators $\hat{A}$

By definition, put $\ell_{A,\psi}(\varphi) := (\partial_\varphi(A))(\psi)$ for any generator $\varphi$ of an evolutionary field $\partial_\varphi$, $\psi \in \Omega$, and a total differential operator $A$. We note that $\ell_{A,\psi}$ is an operator in total derivatives w.r.t. its argument $\varphi$ and hence the adjoint $\ell^*_{A,\psi}$ is well defined.

**Theorem 6** (A criterion of $[\![A, A]\!] = 0$, [5])**.** *A linear skew-adjoint operator $A$ in total derivatives is Hamiltonian if and only if the relation*

$$\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A\big(\ell^*_{A,\psi_2}(\psi_1)\big) \tag{A.1}$$

*holds for all $\psi_1, \psi_2 \in \Omega$. The rhs of (A.1) is skew-symmetric w.r.t. $\psi_1, \psi_2$.*

The proof is based on a straightforward calculation of the value of the variational Schouten bracket $[\![\hat{A}, \hat{A}]\!]$ for a variational Poisson bi-vector $\hat{A}$ on three Hamiltonians $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. Let $\psi_i = \delta\mathcal{H}_i/\delta u \in \Omega$ be the respective variational covectors. The Jacobi identity $[\![\hat{A}, \hat{A}]\!](\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) = 0$ can be expressed as $\langle \mathbf{b}(\psi_1, \psi_2), \psi_3\rangle = 0$, where $\mathbf{b}$ is a differential operator w.r.t. each argument and $\langle , \rangle$ is the coupling on $\widehat{\Omega} \times \Omega$ that takes values in the space of the Hamiltonians. Since $\psi_3 \in \Omega$ is arbitrary, we have that $\mathbf{b}(\psi_1, \psi_2) = 0$ for all $\psi_1, \psi_2 \in \Omega$. The calculation shows that $\mathbf{b}(\psi_1, \psi_2)$ is equal to the left-hand side minus the right-hand side of (A.1), which yields (5).

**Proof of theorem 6.** The substitution principle [4] implies that it suffices to verify the Jacobi identity $J(\psi_1, \psi_2, \psi_3) = 0$ for the elements $\psi_i \in \operatorname{im}\mathbf{E}$ in the image of the variational derivative **E**. Let $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ and $\mathcal{H}_\gamma$ be Hamiltonians. The Jacobi identity is

$$\begin{aligned}
&\{\{\mathcal{H}_\alpha, \mathcal{H}_\beta\}_A, \mathcal{H}_\gamma\}_A + \{\{\mathcal{H}_\beta, \mathcal{H}_\gamma\}_A, \mathcal{H}_\alpha\}_A + \{\{\mathcal{H}_\gamma, \mathcal{H}_\alpha\}_A, \mathcal{H}_\beta\}_A \\
&= -\sum_{\circlearrowleft} \partial_{A(\psi_\gamma)}(\langle A(\psi_\alpha), \psi_\beta\rangle) \\
&= -\sum_{\circlearrowleft} \big[\big\langle \partial_{A(\psi_\gamma)}(A)(\psi_\alpha), \psi_\beta\big\rangle + \big\langle A(\partial_{A(\psi_\gamma)}(\psi_\alpha)), \psi_\beta\big\rangle + \big\langle A(\psi_\alpha), \partial_{A(\psi_\gamma)}(\psi_\beta)\big\rangle\big] = 0.
\end{aligned} \tag{A.2}$$

Consider the elements of the second sum,

$$\begin{aligned}
\big\langle A(\partial_{A(\psi_\gamma)}(\psi_\alpha)), \psi_\beta\big\rangle &= \big\langle \psi_\beta, A(\partial_{A(\psi_\gamma)}(\psi_\alpha))\big\rangle = -\big\langle A(\psi_\beta), \partial_{A(\psi_\gamma)}(\psi_\alpha)\big\rangle \\
&= -\big\langle A(\psi_\beta), \ell_{\psi_\alpha}(A(\psi_\gamma))\big\rangle = -\big\langle \ell^*_{\psi_\alpha}(A(\psi_\beta)), A(\psi_\gamma)\big\rangle \\
&= \big(\text{by } \ell^*_{\mathbf{E}(\mathcal{H}_\alpha)} = \ell_{\mathbf{E}(\mathcal{H}_\alpha)}\big) \\
&= -\big\langle \ell_{\psi_\alpha}(A(\psi_\beta)), A(\psi_\gamma)\big\rangle = -\big\langle A(\psi_\gamma), \ell_{\psi_\alpha} A(\psi_\beta)\big\rangle.
\end{aligned}$$

Substituting this back in (A.2), we obtain

$$\begin{aligned}
0 &= -\sum_{\circlearrowleft} \big\langle (\partial_{A(\psi_\gamma)}(A))(\psi_\alpha), \psi_\beta\big\rangle + \left[\sum_{\circlearrowleft} \big\langle A(\psi_\gamma), \ell_{\psi_\alpha} A(\psi_\beta)\big\rangle - \sum_{\circlearrowleft} \big\langle A(\psi_\alpha), \ell_{\psi_\beta} A(\psi_\gamma)\big\rangle\right] \\
&= -\big\langle (\partial_{A(\psi_\gamma)}(A))(\psi_\alpha), \psi_\beta\big\rangle - \big\langle (\partial_{A(\psi_\alpha)}(A))(\psi_\beta), \psi_\gamma\big\rangle - \big\langle (\partial_{A(\psi_\beta)}(A))(\psi_\gamma), \psi_\alpha\big\rangle.
\end{aligned}$$

Now set $\alpha = 3$, $\beta = 2$, $\gamma = 1$; thence we have

$$0 = -\big\langle (\partial_{A(\psi_1)}(A))(\psi_3), \psi_2\big\rangle - \big\langle (\partial_{A(\psi_3)}(A))(\psi_2), \psi_1\big\rangle - \big\langle (\partial_{A(\psi_2)}(A))(\psi_1), \psi_3\big\rangle. \tag{A.3}$$

Consider the first summand,

$$\begin{aligned}
\big\langle (\partial_{A(\psi_1)}(A))(\psi_3), \psi_2\big\rangle &= \big\langle (\ell_{A,\psi_3}(A(\psi_1))), \psi_2\big\rangle = \big\langle A(\psi_1), \ell^*_{A,\psi_3}(\psi_2)\big\rangle \\
&= \big(\text{by } \ell^*_{A,\psi_1}(\psi_2) = \ell^*_{A^*,\psi_2}(\psi_1)\big) \\
&= \big\langle A(\psi_1), \ell^*_{A^*,\psi_2}(\psi_3)\big\rangle = \big\langle \ell_{A^*,\psi_2}(A(\psi_1)), \psi_3\big\rangle \\
&= -\big\langle \ell_{A,\psi_2}(A(\psi_1)), \psi_3\big\rangle.
\end{aligned} \tag{A.4}$$

Next, the second summand in (A.3) is equal to $\big\langle (\partial_{A(\psi_3)}(A))(\psi_2), \psi_1\big\rangle =$

$$\big\langle \psi_1, \ell_{A,\psi_2}(A(\psi_3))\big\rangle = \big\langle \ell^*_{A,\psi_2}(\psi_1), A(\psi_3)\big\rangle = -\big\langle A(\ell^*_{A,\psi_2}(\psi_1)), \psi_3\big\rangle. \tag{A.5}$$

Now consider the third term on the right-hand side of (A.3),

$$\big\langle (\partial_{A(\psi_2)}(A))(\psi_1), \psi_3\big\rangle = \big\langle (\ell_{A,\psi_1}(A(\psi_2)), \psi_3\big\rangle. \tag{A.6}$$

Substituting (A.4) through (A.6) into (A.3), we finally obtain

$$\big\langle \ell_{A,\psi_2}(A(\psi_1)), \psi_3\big\rangle + \big\langle A(\ell^*_{A,\psi_2}(\psi_1), \psi_3\big\rangle - \big\langle (\ell_{A,\psi_1}(A(\psi_2)), \psi_3\big\rangle = 0,$$

whence follows (A.1). The proof is complete. □

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